$\mathbb{C}^*$- Actions on Stein analytic spaces with isolated singularities

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1 Introduction

Let V be an irreducible complex analytic space of dimension two with normal singularities and $\varphi : \mathbb{C}^* \times V \to V$ a holomorphic action of the group $\mathbb{C}^*$ on V. Denote by $\mathcal{F}_\varphi$ the foliation on V induced by φ. The leaves of this foliation are the one-dimensional orbits of φ. We will assume that there exists a *dicritical* singularity $p \in V$ for the $\mathbb{C}^*$-action, i.e. for some neighborhood $p \in W \subset V$ there are infinitely many leaves of $\mathcal{F}_\varphi|_W$ accumulating only at p. The closure of such a local leaf is an invariant local analytic curve called a *separatrix* of $\mathcal{F}_\varphi$ through p. In [14] Orlik and Wagreich studied the 2-dimensional affine algebraic varieties embedded in $\mathbb{C}^{n+1}$, with an isolated singularity at the origin, that are invariant by an effective action of the form $\sigma_Q(t,(z_0,...,z_n)) = (t^{q_0}z_0,...,t^{q_n}z_n)$ where $Q = (q_0,...,q_n) \in \mathbb{N}^{n+1}$, i.e. all q_i are positive integers. Such actions are called *good* actions. In particular they classified the algebraic surfaces embedded in $\mathbb{C}^3$ endowed with such an action. It is easy to see that any good action on a surface embedded in $\mathbb{C}^{n+1}$ has a dicritical singularity at $0 \in \mathbb{C}^{n+1}$. Conversely, it is the purpose of this paper to show that good actions are the models for analytic $\mathbb{C}^*$-actions on Stein analytic spaces of dimension two with a dicritical singularity. In this paper all spaces are connected and complex analytic.

Theorem 1. *Let V be a normal Stein analytic space of dimension two and φ a $\mathbb{C}^*$-action on V with at least one dicritical singularity $p \in V$. There is an embedding $h : V \to \mathbb{C}^{n+1}$, for some n, onto an algebraic subvariety $\mathcal{V} := h(V)$ and a good action σ_Q on $\mathbb{C}^{n+1}$, leaving $\mathcal{V}$ invariant and analytically conjugate to φ, i.e.,*

$$h(\varphi(t,x)) = \sigma_Q(t,h(x)),\ \forall x \in V,\ t \in \mathbb{C}^*.$$

Notice that this theorem implies that there is no other singularity of φ apart from $p \in V$. The above theorem can be considered as a GAGA principle for Stein varieties with $\mathbb{C}^*$-actions. This answers a question posed by some authors (see for instance the comments after Proposition 1.1.3 in [14] and references there).

Corollary 1. *Let V be a smooth Stein surface endowed with a $\mathbb{C}^*$-action having a dicritical singularity at $p \in V$. Then V is biholomorphic to $\mathbb{C}^2$.*

The proof of Theorem 1 will also provide a proof of the following:

Theorem 2. *The moduli space of pairs (V, φ), $dim(V) = 2$, with at least one dicritical singularity for φ as in Theorem 1, is the following data*

1. *A Riemann surface σ_0 of genus g and s-points $r_1, r_2, \ldots, r_s$ on σ_0 considered up to the automorphism group of σ_0.*

2. *A line bundle L on σ_0 with $c(L) = -k \leq -1$.*

3. *For each $i = 1, 2, \ldots, s$ a sequence of integers $-k^i_j$, $j = 1, 2, \ldots, n_i$, $k^i_j \geq 2$, such that*
$$\sum_{i=1}^{s} \frac{1}{[k^i_1, k^i_2, \ldots, k^i_{n_i}]} < k,$$
where
$$[k^i_1, k^i_2, ..., k^i_{n_i}] = k^i_1 - \frac{1}{k^i_2 - \frac{1}{\ddots}}.$$

Conversely, 1, 2 and 3 imply the existence of a pair (V, φ).

The above data can be read from the minimal resolution of the desingularization at $p \in V$ of the foliation induced by φ.

The proof of Theorem 1 consists of the following steps. We first analize in §2 the resolution of the singularity $p \in V$ and obtain Theorem 3 which is an analytic version of a theorem proved in [14]. It turns out that there is only one element σ_0 of arbitrary genus in the divisor of the resolution of $p \in V$ on which $\mathbb{C}^*$ acts transversely. All other divisors are Riemann spheres and are invariant under the action of $\mathbb{C}^*$. In §3 we linearize the $\mathbb{C}^*$-action in a neighborhood of σ_0. The main theorem of this section, Theorem 4, does not require any hypothesis on the self intersection number of σ_0. In §4 we first introduce

the linear model for the resolution of $p \in V$ and then extend the linearization obtained in the previous section to the basin of attraction of $p \in V$. In §5 we prove that the basin of attraction of $p \in V$ is the whole space V and so the constructed linearization provides the conjugacy claimed in Theorem 1.

2 Resolution of singularities

In order to prove Theorem 1 we first describe the resolution of the action φ and then compare it with the resolution of a model good action.

2.1 Holomorphic foliations

We start with the resolution theorem for normal two dimensional singularities (see [9]) and the resolution theorem for holomorphic foliations (see [17], [5]) that combined together assert, first, that there exists a proper holomorphic map $\rho : \tilde{V} \to V$ such that $D{:=}\rho^{-1}(p) = \bigcup_{i=0}^{r} \sigma_i$, is a finite union of compact Riemann surfaces σ_i intersecting at most pairwise at normal crossing points, and then that $\tilde{V}$ is an analytic space of dimension two with no singularities near D. More precisely, the σ_i's are compact Riemann surfaces without singularities such that if $\sigma_i \cap \sigma_j \neq \emptyset$ then σ_i and σ_j have normal crossing and $\sigma_i \cap \sigma_j \cap \sigma_k = \emptyset$ if $i \neq j \neq k \neq i$. Moreover, the *intersection matrix* $(\sigma_i \cdot \sigma_j)$ is negative definite ([9])and the restriction of ρ to $\tilde{V} \backslash D$ is a biholomorphism onto $V \backslash \{p\}$. By means of this restriction $\mathcal{F}_\varphi$ induces a foliation $\tilde{\mathcal{F}}_\varphi$ on $\tilde{V} \backslash D$ that can be extended to $\tilde{V}$ as a foliation with isolated singularities. Each one of these singularities can be written in local coordinates (x, y) around $0 \in \mathbb{C}^2$ in one of the following forms : (i) *simple singularities*: $xdy - y(\mu + \cdots)dx = 0$, $\mu \notin \mathbb{Q}_+$, where the points denote higher order terms; (ii) *saddle-node singularities*: $x^{m+1}dy - (y + ax^m y + \cdots)dx = 0$, $a \in \mathbb{C}$, $m \in \mathbb{N}$. A simple singularity has two invariant manifolds crossing normally, they correspond to the x and y-axes. The saddle-node has an invariant manifold corresponding to the y-axis and, depending on the higher order terms, it may not have another invariant curve (see [11]). The resolution of $\mathcal{F}_\varphi$ can be obtained in such a way that the elements σ_i fall in two categories. Either σ_i is a *dicritical component*, when $\tilde{\mathcal{F}}_\varphi$ is everywhere transverse to σ_i, or a *nondicritical* component when σ_i is tangent to $\tilde{\mathcal{F}}_\varphi$. In a similar way, by means of the restriction ρ to $\tilde{V} \backslash D$ the $\mathbb{C}^*$- action φ on $V \backslash \{p\}$ induces a $\mathbb{C}^*$- action $\tilde{\varphi}$ on $\tilde{V} \backslash D$ that can be extended to D as a $\mathbb{C}^*$- action (see [15]). For

this it is enough to observe that $D \subset \tilde{V}$ is analytic of codimension one, $\tilde{V}$ is a normal analytic space and $\tilde{\varphi}$ is bounded in a neighborhood of D. We have therefore that the orbits of $\tilde{\varphi}$ are contained in the leaves of the foliation $\tilde{\mathcal{F}}_\varphi$.

The divisor D forms a graph with vertices σ_i and sides the nonempty intersections $\sigma_i \cap \sigma_j$. A *star* is a contractible connected graph where at most one vertex, called its *center*, is connected with more than two other vertices. A *weighted graph* is a graph where at each vertex is associated its genus and its self-intersection number.

2.2 On a theorem of Orlik and Wagreich

In this section we describe the resolution of p as a singular point of V and as a singularity of $\mathcal{F}_\varphi$.

Theorem 3. *Let V be a normal Stein analytic space of dimension two and φ a $\mathbb{C}^*$-action on V with a dicritical singularity at $p \in V$. Then there is a resolution $\rho : \tilde{V} \to V$ of $\mathcal{F}_\varphi$ at the point $p \in V$ such that*

1. *$\rho^{-1}(p) = \bigcup_{i=0}^r \sigma_i$ is a weighted star graph centered at the Riemann surface σ_0 of genus g, and consisting of Riemann spheres σ_i, $i > 0$;*
2. *σ_0 is the unique dicritical component of $\tilde{\mathcal{F}}_\varphi = \rho^* \mathcal{F}_\varphi$;*
3. *the pull-back action $\tilde{\varphi}$ on $\tilde{V}$ is trivial on σ_0 and nontrivial on each σ_i, $i > 0$;*
4. *The singular points of $\tilde{\mathcal{F}}_\varphi$ are $\sigma_i \cap \sigma_j \neq \emptyset$, $i, j \neq 0$ and all of them are simple.*

In the algebraic context in which V is affine and the $\mathbb{C}^*$-action is algebraic, the above theorem with items *1*, *2* and *3* is a result of Orlik and Wagreich (see [14]). Our proof uses the theory of holomorphic foliations on complex manifolds instead of topological methods. In order to prove Theorem 3 we need the following index theorem.

2.3 The Index theorem

Let σ be a Riemann surface embedded in a two dimensional manifold S ; $\mathcal{F}$ a foliation on S which leaves σ invariant and $q \in \sigma$. There is a neighborhood of q where σ can be expressed by $(f = 0)$ and $\mathcal{F}$ is induced by the holomorphic 1-form ω written as $\omega = h df + f\eta$. Then we can associate the following index:

$$i_q(\mathcal{F}, \sigma) := -\text{Residue}_q(\frac{\eta}{h})|_\sigma$$

relative to the invariant submanifold σ. In the case of a simple singularity as defined above if σ is locally $(y = 0)$ and $q = 0$, this index is equal to μ (quotient of eigenvalues). In the case of a saddle-node, if σ is equal to $(x = 0)$ and $q = 0$, this index is zero. At a regular point q of $\mathcal{F}$ the index is zero. The index theorem of [5] asserts that the sum of all the indices at the points in σ is equal to the self-intersection number $\sigma \cdot \sigma$:

$$\sum_{q \in \sigma} i_q(\mathcal{F}, \sigma) = \sigma \cdot \sigma.$$

2.4 Proof of Theorem 3

By hypothesis, in the resolution of $p \in V$ there is at least one dicritical component, say σ_0. Then the action $\tilde{\varphi}$ extends to σ_0 as a set of fixed points. We claim that σ_0 is the unique dicritical component. Indeed, at each dicritical component the $\mathbb{C}^*$- action $\tilde{\varphi}$ is trivial. Since V is normal at $p \in V$, $\rho^{-1}(p)$ is connected ([9]), thus if there is another dicritical component, say σ_i, then there would exists $\mathbb{C}^*$- orbits of $\tilde{\varphi}$, with compact analytic closure crossing σ_0 and σ_i transversely contradicting the fact that V is Stein. Thus σ_0 is the only dicritical component, and the action $\tilde{\varphi}$ is trivial on σ_0. The same argument shows that there cannot be cycles of components of D starting and ending at σ_0. Thus the graph associated to ρ is contractible.

A *linear chain* at a point $q \in \sigma_0$ is a union of compact Riemann surfaces, elements of the divisor D, say $\sigma_1, ..., \sigma_n$ such that $\sigma_1 \cap \sigma_0 = \{q\}$ and $\sigma_i \cap \sigma_j$ is nonempty if and only if $i = j - 1$ and in this case it is a point, for $j = 2, ..., n$.

Lemma 1. *Suppose that $r_1, r_2, ..., r_s$ are the crossing points at σ_0 of the divisor D. Then the divisor D consists of the union of σ_0 and linear chains of Riemann spheres at each of these crossing points.*

Proof. Consider the divisor D at the point r_1 renamed as p_0. Let σ_1 be such that $p_0 = \sigma_0 \cap \sigma_1$. We claim that the $\mathbb{C}^*$-action $\tilde{\varphi}$ on σ_1 is nontrivial with a fixed point at p_0. Indeed it can be represented in local coordinates (x, y), where $(x = 0) = \sigma_0$, $(y = 0) = \sigma_1$, by the vector field $Y = (Y_1, 0)$ with $Y_1(0, y) = 0$. Consider the restriction of the action $\tilde{\varphi}$ to the subgroup $S^1 \subset \mathbb{C}^*$. Then in the $\mathbb{C}$-plane $(y = y_0)$ the S^1-orbit of a generic point (x, y_0), $x \neq 0$, will turn l times around $(0, y_0)$ and this number, which is different from

zero, will be constant as $y_0 \to 0$. Therefore $\tilde{\varphi}$ extends to the x-axis σ_1 as a nontrivial $\mathbb{C}^*$-action. Therefore σ_1 is a Riemann sphere and there is another point $p_1 \in \sigma_1$ which is fixed by $\tilde{\varphi}$. Since p_1 is the unique singularity of $\tilde{\mathcal{F}}_\varphi$ in σ_1 we must have that the index of $\tilde{\mathcal{F}}_\varphi$ with respect to the invariant manifold σ_1 at p_1 is given by ([5])

$$i_{p_1}(\tilde{\mathcal{F}}_\varphi, \sigma_1) = \sigma_1.\sigma_1 = -k_1, \; k_1 \in \mathbb{N}.$$

Therefore p_1 cannot be a saddle-node, as in this case this index would be zero. This implies that p_1 is simple for $\tilde{\mathcal{F}}_\varphi$. Either the chain ends at σ_1 or there is another component, say σ_2, such that $\{p_1\} = \sigma_1 \cap \sigma_2$. In this last case, p_1 is simple. We claim that the action $\tilde{\varphi}$ on σ_2 is nontrivial. Indeed, let (x, y) be a system of coordinates in a neighborhood $\mathcal{N}$ of $p_1 = (0, 0)$ such that $(x = 0) = \sigma_1 \cap \mathcal{N}$, $(y = 0) = \sigma_2 \cap \mathcal{N}$. By derivation along the parameter of the group, the action φ induces a vector field Y on $\mathcal{N}$. Assuming by contradiction that φ is trivial on σ_2 we would have $Y(x, 0) = 0$ and we can assume, changing coordinates if necessary, that $DY(x, 0) = diag(0, \lambda_x)$, $\lambda_0 \neq 0$. By continuity, $\lambda_x \neq 0$ for x small enough. By the invariant manifold theorem for ordinary differential equations, there is a fibration invariant by Y, transverse to σ_2, whose fibers are the subsets of $\mathcal{N}$ defined as $\tau_x = \{(x, y); \lim_{t\to 0} \varphi(t, (x, y)) = (x, 0)\}$, $\tau_0 = \sigma_1$. Thus σ_2 is a dicritical component of $\tilde{\mathcal{F}}_\varphi$, which is a contradiction. Therefore σ_2 will be a *Riemann sphere* with another fixed point $p_2 \in \sigma_2$ for the action $\tilde{\varphi}$. It is clear that the corresponding index will be given by

$$i_{p_2}(\tilde{\mathcal{F}}_\varphi, \sigma_2) = -k_2 + 1/k_1 \neq 0, \; k_2 = -\sigma_2.\sigma_2 \in \mathbb{N}.$$

More generally, the linear chain will consist of a finite sequence of elements of the divisor $\sigma_0, \sigma_1, ..., \sigma_n$ such that σ_i, for $i \neq 0$, is a Riemann sphere where the action $\tilde{\varphi}$ is nontrivial, and $\sigma_i \cap \sigma_{i+1} = \{p_i\}$ is a simple singularity of $\tilde{\mathcal{F}}_\varphi$ for $i = 1, ..., n-1$. Denote by $-k_i = \sigma_i.\sigma_i, k_i \in \mathbb{N}$. At each point p_i the index of this singularity relative to σ_n is

$$i_{p_j}(\tilde{\mathcal{F}}_\varphi, \sigma_j) = -[k_j, k_{j-1}, ..., k_1],$$

where we have a continued fraction

$$[k_j, k_{j-1}, ..., k_1] = k_j - \cfrac{1}{k_{j-1} - \cfrac{1}{\ddots}}.$$

We claim that the numbers $[k_j, k_{j-1}, ..., k_1]$, $j = 1, ..., n$, are all well defined and different from zero. Indeed, this is a consequence of the fact that the intersection matrix $(\sigma_i \cdot \sigma_i)$

is negative definite ([9]). Let M be a real symmetric $n \times n$ matrix and Q a non-singular real $n \times n$ matrix. Then M is negative definite if and only if $Q^t M Q$ is negative definite. Given the matrix $M = (\sigma_i \cdot \sigma_j)$ we take Q as the matrix with one's in the diagonal, a in the $(1,2)$ entry, and zeros elsewhere. Then a convenient choice of a will yield a matrix $Q^t M Q$ with $-k_1$ in the $(1,1)$ entry and zeros in the $(1,2)$ and $(2,1)$ entries. Repeating this procedure we obtain that the following diagonal matrix

$$\operatorname{diag}(-k_1, -[k_2, k_1], ..., -[k_n, k_{n-1}, ..., k_1])$$

is negative definite, proving the claim and the lemma.

□

Theorem 3 follows from the above discussion and Lemma 1.

3 Linearization around the dicritical divisor

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. In the previous section we saw that the multiplicative pseudo group $\mathcal{G} = (\mathbb{C}, \mathbb{D}) - \{0\}$ acts on $(\tilde{V}, \sigma_0)$ and the flow of the action φ is transverse to σ_0. The purpose of this section is to show that such an action is biholomorphically conjugated with the canonical $\mathcal{G}$-action on the normal bundle to σ_0 in $\tilde{V}$.

3.1 $\mathcal{G}$-transverse actions to a Riemann surface

Let σ be a Riemann surface embedded in a surface S. We say that ψ is a transverse $\mathcal{G}$-action on (S, σ) if

1. For all $a \in \sigma$ and $t \in \mathcal{G}$ we have $\psi(t, a) = a$.

2. There is a foliation $\mathcal{F}$ on (S, σ), transverse to σ such that each leaf of $\mathcal{F}$ is the closure of $\{\psi(t, a) \mid t \in \mathcal{G}\}$ for some $a \in (S, \sigma) - \sigma$.

A typical example of a $\mathcal{G}$-action is the following: We consider a line bundle L on σ and the embedding $\sigma \hookrightarrow L$ given by the zero section. Now for every $q \in \mathbb{N}$ we have a transverse $\mathcal{G}$-action on (L, σ) given by $(t, a) \mapsto t^q a$. It turns out that up to biholomorphy these are the only transverse $\mathcal{G}$-actions.

Theorem 4 (Linearization theorem). *Let σ be a Riemann surface embedded in a surface S and ψ a transverse $\mathcal{G}$-action on (S, σ). Then ψ is linearizable in the sense that there exist a biholomorphism $h : (S, \sigma) \to (N, \sigma)$, where N is the normal bundle to σ in S, and a natural number q such that $h(\psi(t, a)) = t^q h(a)$ for any $a \in (S, \sigma)$.*

Notice that the linearization of ψ yields also the linearization of the associated foliation. An immediate corollary of the above theorem is that non-linearizable neighborhoods do not admit any transversal $\mathcal{G}$-action. For instance, Arnold's example in which σ is a torus of self-intersection number zero in some complex manifold of dimension two is not linearizable and so it does not admit any transversal $\mathcal{G}$-action (see [2]).

3.2 Local linearization

Let $S = (\mathbb{C}^2, 0)$ and $0 \in \sigma \subset S$ be a smooth curve in S. In a similar way as before we define a $\mathcal{G}$-action on (S, σ) transverse to σ and call it the local transverse $\mathcal{G}$-action.

Lemma 2. *Any local transverse $\mathcal{G}$-action can be written in a local system of coordinates in the form $\psi(t, (x, y)) = (x, t^q y)$.*

Proof. We take a coordinates system (x, y) around $0 \in \mathbb{C}^2$ such that the the foliation $\mathcal{F}_\psi$ is given by $dx = 0$ and σ is given by $y = 0$. In these coordinates the flow ψ_t of the $\mathbb{C}^*$-action is given by:

$$\psi_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0), \ \psi_t(x, y) = (x, p_{t,x}(y)).$$

Since the orbits of ψ tend to σ when t tends to zero, $p_{t,x}$ is a holomorphic function in $t \in (\mathbb{C}, \mathbb{D})$. We have also $p_{t,x}(0) = 0$ because σ is the set of fixed points of ψ. We can write $p_{t,x}(y)$ as a series

$$p_{t,x}(y) = \sum_{i=1} p_i(t, x) y^i.$$

Substituting the above term in $\psi(t_1 t_2, a) = \psi(t_1, \psi(t_2, a))$ we obtain

$$p_1(t_1 t_2, x) = p_1(t_1, x) p_1(t_2, x), \ t_1, t_2 \in \mathcal{G}, \ x \in (\mathbb{C}, 0).$$

Since p_1 is holomorphic at $t = 0$, the derivation of the above equality in t_1 implies that $p_1(t, x) = t^q$ for some $q \in \mathbb{N}$. Now, by the Theorem on the linearization of germs of holomorphic mapings, there is a unique $f_{t,x} : (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$ which is tangent to the identity, depends holomorphically on t, x and

$$f_{t,x}^{-1} \circ p_{t,x} \circ f_{t,x}(y) = t^q y.$$

The $\mathbb{C}^*$-action ψ in the coordinates $(\tilde{x}, \tilde{y}) = (x, f_{t,x}(y))$ has the desired form. $\square$

Now consider on S a foliation $\mathcal{F}$ which is transverse to σ (no $\mathcal{G}$-action is considered). Let ω be a 1-form on S such that

$$div(\omega) = \sigma + nL_0,$$

where $n \in \mathbb{Z}$ and L_0 is the leaf of $\mathcal{F}$ through $0 \in S$.

Lemma 3. *Given a local system of coordinates x in σ, there is a unique system of coordinates $(\tilde{x}, \tilde{y})$ in S such that*

1. *The restriction of $\tilde{x}$ to σ is x;*

2. *The 1-form ω in $(\tilde{x}, \tilde{y})$ is of the form $\tilde{x}^n \tilde{y} d\tilde{x}$.*

Proof. For the proof of the existence we take a coordinates system $(\tilde{x}, \tilde{y})$ in a neighborhood of 0 in S such that σ and $\mathcal{F}$ in this coordinate system are given respectively by $\tilde{y} = 0$ and $d\tilde{x} = 0$ and $\tilde{x} \mid_\sigma = x$. We write $\omega = p\tilde{x}^n \tilde{y} d\tilde{x}$, where $p \in \mathcal{O}_S,\ p(0) \neq 0$. By changing the coordinates $(\tilde{x}, \tilde{y}) \to (\tilde{x}, p\tilde{y})$ we obtain the desired coordinate system. The uniqueness follows from the fact that any local biholomorphism $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ which is the identity in $\tilde{y} = 0$ and $f^* \tilde{x}^n \tilde{y} d\tilde{x} = \tilde{x}^n \tilde{y} d\tilde{x}$ is the identity map. $\square$

3.3 Construction of differential forms

Consider a Riemann surface σ embedded in a two dimensional manifold S. We take a meromorphic section s of the normal bundle N of σ in S and set

$$div(s) = \sum n_i p_i,\ n_i \in \mathbb{Z},\ p_i \in \sigma.$$

Lemma 4. *For a transverse $\mathcal{G}$-action ψ on (S, σ), there is a meromorphic function u on (S, σ) such that*

1. $$div(u) = \sigma - \sum n_i p_i,\ n_i \in \mathbb{Z},\ p_i \in \sigma,$$

2. $$u(\psi(t, a)) = t^q u(a),\ a \in (S, \sigma),\ t \in \mathcal{G}.$$

Let $\tilde{v}$ be an arbitrary meromorphic function on σ and v its extension to S along the foliation $\mathcal{F}$. The 1-form

$$\omega = udv$$

has the properties:

1. *ω induces the foliation $\mathcal{F}$;*

2. *The divisor of ω is $\sigma + K$, where K is $\mathcal{F}$-invariant.*

3. *$\psi_t^* \omega = t^q \omega, \ t \in \mathcal{G}, \ where \ \psi_t(x) = \psi(t, x)$.*

Proof. In a local coordinate system (x_α, y_α) in a neighborhood U_α of a point p_α of σ in S one can write the $\mathcal{G}$-action as follows

$$\psi(t, (x_\alpha, y_\alpha)) = (x_\alpha, t^q y_\alpha),$$

where $\sigma \cap U_\alpha = \{y_\alpha = 0\}$. The meromorphic function $u_\alpha = x_\alpha^{-n} y_\alpha$, where $n = n_i$ if $p = p_i$ for some i and $n = 0$ otherwise, satisfies the conditions *1, 2* in U_α. We define $u_{\alpha\beta} := \frac{u_\alpha}{u_\beta}$. Now $L := \{u_{\alpha\beta}\} \in H^1(S, \pi^{-1}\mathcal{O}_\sigma^*) = H^1(\sigma, \mathcal{O}_\sigma^*)$, where $\pi : S \to \sigma$ is the projection along the fibers. On the other hand, the line bundle associated to σ in S and then restricted to σ is the normal bundle of σ in S and so by definition L restricted to σ is the trivial bundle. This means that there are $a_\alpha \in \pi^{-1}\mathcal{O}_\sigma^*(U_\alpha)$ such that $u_{\alpha\beta} = \frac{a_\alpha}{a_\beta}$. Now, $\frac{u_\alpha}{a_\alpha}$ define a meromorphic function on S with the desired properties. □

Remark 1. In the case in which we have a transverse foliation $\mathcal{F}$ without any transverse ψ action, the linearization of $\mathcal{F}$ requires $\sigma \cdot \sigma < min(2 - 2g, 0)$, where g is the genus of σ (see [3, 4]). In this case, in order to construct u with the first property we used this hypothesis and proved that the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(\sigma)$ is injective. As we saw in the proof of Lemma 4, in the presence of a transverse $\mathcal{G}$-action we do not need any hypothesis on $\sigma \cdot \sigma$.

3.4 Holomorphic equivalence of neighborhoods

Now we consider two embeddings of σ with transverse foliations.

Lemma 5. *Let σ be a Riemann surface embedded in two surfaces $S_i, \ i = 1, 2$ and let $\mathcal{F}_i$ be a foliation transverse to σ on S_i induced by a 1-form ω_i such that the divisor of ω_i is*

$\sigma + K_i$, where K_i is $\mathcal{F}_i$-invariant and K_1 and K_2 restricted to σ coincide. Then there is a unique biholomorphism $h : (S_1, \sigma) \to (S_2, \sigma)$ such that $h^*\omega_2 = \omega_1$.

Proof. Using Lemma 3 we conclude that for a point $a \in \sigma$ there is a unique $h : (S_1, \sigma, a) \to (S_2, \sigma, a)$ such that h restricted to σ is the identity map and $h^*\omega_2 = \omega_1$. The uniqueness implies that these local biholomorphisms coincide in their common domains and so they give us a global biholomorphism $h : (S_1, \sigma) \to (S_2, \sigma)$ with the desired property.

$\square$

3.5 Proof of the linearization theorem

Let us now prove Theorem 4. Take $i = 1, 2$. Let σ be a Riemann surface embedded in two surfaces S_i and let ψ_i be a transverse $\mathcal{G}$-action on (S_i, σ) with the multiplicity q and corresponding foliation $\mathcal{F}_i$. By Lemma 4 we can construct a 1-form ω_i with the properties *1, 2, 3.* By construction of ω_i, if $div(\omega_i) = \sigma + K_i$ then K_i restricted to σ depends only on $\tilde{v}$ and s and so we can take the K_i's so that $K_1 \mid_\sigma = K_2 \mid_\sigma$. Now Lemma 5 implies that there is a unique biholomorphism $h : (S_1, \sigma) \to (S_2, \sigma)$ such that $h^*\omega_2 = \omega_1$. We claim that h conjugates also the ψ_i's. Fix $t \in \mathcal{G}$ and let $\psi_{i,t} : (S_i, \sigma) \to (S_i, \sigma)$ be a biholomorphism defined by

$$\psi_{i,t}(a) := \psi_i(t, a), \; a \in (S_i, \sigma).$$

We have

$$h^*\psi_{2,t}^*\omega_2 = h^*t^q\omega_2 = t^q\omega_1 = \psi_{1,t}^*\omega_1 = \psi_{1,t}^*h^*\omega_2.$$

Since by Lemma 5 the sole $f : (S_2, \sigma) \to (S_2, \sigma)$ such that $f^*\omega_2 = \omega_2$ is the identity map, we conclude that $h^*\psi_{2,t}^* = \psi_{1,t}^*h^*$ and so $h(\psi_1(t, a)) = \psi_2(t, h(a))$.

4 Linearization in the attraction basin

In this section we associate to the foliation $\tilde{\mathcal{F}}_\varphi$ a *linear model* and prove a linearization result based on the existence of the $\mathcal{G}$-action transverse to σ_0.

4.1 The linear model

We can associate to the pair $(\tilde{\mathcal{F}}_\varphi, \tilde{V})$ a linear model constructed as follows. Let L be the normal bundle of σ_0 in $\tilde{V}$. We denote by L^{-1} the dual of L. We can glue L and L^{-1}

together and obtain a compact projective manifold $\bar{L}$ in the following way: Let $\{U_\alpha\}_{\alpha\in I}$ be an open covering of σ_0 and z_α (resp. z'_α) a holomorphic without zero section of L (resp. L^{-1}) on U_α. Then

$$z_\alpha = g_{\alpha\beta} z_\beta,\ z'_\alpha = g_{\alpha\beta}^{-1} z'_\beta,\ L = \{g_{\alpha\beta}\}_{\alpha,\beta\in I} \in H^1(S, \mathcal{O}^*).$$

For a point $a \in L_p, p \in U_\alpha,\ a \neq 0_p$ we define the point $\frac{1}{a} \in L_p^{-1}$ by setting

$$\frac{1}{a} = \frac{z_\alpha(p)}{a} z'_\alpha(p).$$

The map $a \to 1/a$ does not depend on the chart U_α and gives us a biholomorphism between $L - \sigma_0$ and $L^{-1} - \sigma_\infty$, where σ_0 (resp. σ_∞) is the zero section of L (resp. L^{-1}).

For each point $r_i^0 = r_i \in \sigma_0,\ i = 1, 2, \ldots, s$ we denote by r_i^∞ the unique intersection point of σ_∞ and $\bar{L}_{r_i^0}$. By various blow ups starting from r_i^∞ in the chain $\sigma_0, \bar{L}_{r_i^0}, \sigma_\infty$, we can create a chain of divisors

$$\sigma_0, \sigma_1^i, \sigma_2^i, \ldots, \sigma_{n_i}^i, \tilde{\sigma}, \tau_{m_i}^i, \tau_{m_i-1}^i, \ldots, \tau_1^i, \sigma_\infty$$

such that

$$\sigma_j^i \cdot \sigma_j^i = -k_j^i,\ j = 1, 2, \ldots, n_i, \quad \tilde{\sigma} \cdot \tilde{\sigma} = -1, \quad -l_j^i := \tau_j^i \cdot \tau_j^i < -1, j = 1, 2, \ldots, m_i.$$

The chain of self-intersections of the divisors in the blow-up process is given by:

$$(-k, 0, k), (-k, -1, -1, k-1), (-k, -2, -1, -2, k-1), \ldots, (-k, -k_1^i, -1, \underbrace{-2, \cdots, -2}_{k_1^i - 1 \text{ times}}, k-1)$$

$$(-k, -k_1^i, -2, -1, -3, \underbrace{-2, \cdots, -2}_{k_1^i - 2 \text{ times}}, k-1), \cdots, (-k, -k_1^i, -k_2^i, \cdots, -k_{n_i}^i, -1, l_{m_i}^i, \cdots, l_2^i, l_1^i, k-1).$$

Repeating this construction at each point r_i, $i = 1, ..., s$ we obtain a surface X. Let

$$D_\infty = \sigma_\infty + \sum_{i=1}^{s} \sum_{j=1}^{m_i} \tau_j^i,\ D_0 = \sigma_0 + \sum_{i=1}^{s} \sum_{j=1}^{n_i} \sigma_j^i.$$

Now, $\tilde{\mathcal{V}} := X - D_\infty$ is the desired linear model variety. In $\bar{L}$ we have a canonical $\mathbb{C}^*$ action whose orbits are the fibers of L. It gives us a $\mathbb{C}^*$-action $\tilde{\lambda}$ on $\tilde{\mathcal{V}}$. We denote by $\tilde{\mathcal{F}}_\lambda$ the associated foliation on $\tilde{\mathcal{V}}$. The pair $(\tilde{\mathcal{V}}, \tilde{\mathcal{F}}_\lambda)$ will be called the *linear approximation* of $(\tilde{V}, \tilde{\mathcal{F}}_\varphi)$.

In order to proceed with our discussion we need some definitions: A divisor $Y = \sum_{i=1}^{l} Y_i$ in a two-dimensional surface X is a support of a divisor with positive (resp.

negative) normal bundle if there is a divisor $\tilde{Y} := \sum_{i=1}^{l} a_i Y_i$, where the $a_i,\ i = 1, 2, \ldots, l$ are positive integers, such that $\tilde{Y} \cdot Y_j > 0($ resp. $< 0)$, for $j = 1, \ldots, l$.

We say that the normal bundle of the divisor $\tilde{Y}$ in X is positive (resp. negative). Observe that the normal bundle N of a divisor is positive (resp. negative) if and only if N restricted to each irreducible component of the divisor is positive (resp. negative) (see [8] Proposition 4.3). In fact the above number is the Chern class of $N\mid_{Y_i}$ (see [9] p. 62).

Lemma 6. *The following assertions are equivalent:*

1. *The divisor D_∞ is a support of divisor with positive normal bundle.*

2. *The self-intersection matrix of D_0 is negative definite.*

3. $$\sum_{i=1}^{s} \frac{1}{[k_1^i, k_2^i, \ldots, k_{n_i}^i]} < k.$$

4. $$\sum_{i=1}^{s} \frac{1}{[l_1^i, l_2^i, \ldots, l_{m_i}^i]} > s - k.$$

Proof. *1 ⇒ 2*. From [8] Theorem 4.2 it follows that one can make a blow down of the divisor D_0 and so the self intersection matrix of D_0 is negative definite.

2 ⇒ 3. We remark that the diagonalization of the intersection matrix of D_0 by the procedure given in Lemma 1 leads to

$$\mathrm{diag}(\ldots, -k_{n_i}^i, -[k_{n_i-1}^i, k_{n_i}^i], ..., -[k_1^i, k_2^i, ..., k_{n_i}^i], \ldots, -k + \sum_{i=1}^{s} \frac{1}{[k_1^i, k_2^i, \ldots, k_{n_i}^i]}).$$

Recall that $k_j^i > 1$ for $i = 1, \ldots, s; j = 1, \ldots, n_i$.

3 ⇒ 4. Using the index theorem we have

$$\frac{1}{[k_{n_i}^i, k_{n_i-1}^i, \ldots, k_1^i]} + \frac{1}{[l_{m_i}^i, l_{m_i-1}^i, \ldots, l_1^i]} = 1.$$

Notice that the order of the continued fraction is the inverse of the one we need. However we have that: if

$$-k, -k_1^i, -k_2^i, \ldots, -k_{n_i}^i, -1, -l_{m_i}^i, -l_{m_i-1}^i, \ldots, -l_1^i, k-1$$

is obtained by blow-ups as we explained then

$$-k, -k_{n_i}^i, -k_{n_i-1}^i, \ldots, -k_1^i, -1, -l_1^i, -l_2^i, \ldots, -l_{m_i}^i, k-1$$

is also obtained by blow-ups. This can be proved by induction on the number of blow-ups. Notice that to create each branch of the star we have done only one blow-up centered at a point of σ_∞ (the first blow-up) and so after obtaining the desired star the self intersection of σ_0 is $k-s$.

4 $\Rightarrow$ *1.* We are looking for natural numbers a and $a^i_j,\ j=1,2,\ldots,m_i,\ i=1,2,\ldots,s$ such that the normal bundle of $\tilde{Y}=a\sigma_\infty+\sum_{i=1}^{s}\sum_{j=1}^{m_i}a^i_j\tau^i_j$ is ample, i.e $\tilde{Y}\cdot\sigma>0$ for $\sigma=\sigma_\infty$ and all σ^i_j. These inequalities are translated into:

$$-l^i_ja^i_j+a^i_{j-1}+a^i_{j+1}>0, a^i_0:=n,\ a^i_{m_i+1}:=0,$$

$$a(k-s)+\sum_{i=1}^{s}a^i_{m_i}>0.$$

We rewrite these inequalities in the following way:

$$\frac{a}{a^i_1}>[l^i_1,\frac{a^i_1}{a^i_2}]>\ldots>[l^i_1,l^i_2,\ldots,l^i_{m_i-1},\frac{a^i_{m_i-1}}{a^i_{m_i}}]>[l^i_1,l^i_2,\ldots,l^i_{m_i-1},l^i_{m_i}],$$

$$\sum_{i=1}^{s}\frac{1}{\frac{a}{a^i_1}}>s-k.$$

The existence of positive rational numbers $\frac{a^i_j}{a^i_{j-1}}$ follows from the hypothesis 4. Notice that l^i_j are all greater than 1 and so the $[l^i_1,l^i_2,\ldots,l^i_{m_i-1},l^i_{m_i}]$'s are positive. $\square$

We denote by $\mathcal{V}$ the variety obtained by the blow down of the divisor D_0 in $\tilde{\mathcal{V}}$. We also denote by λ the $\mathbb{C}^*$-action on $\mathcal{V}$ corresponding to $\tilde{\lambda}$ in $\tilde{\mathcal{V}}$.

Proposition 1. *The variety $\mathcal{V}$ is affine algebraic and the $\mathbb{C}^*$- action λ is given by a good action in some affine coordinates.*

Proof. Since the self intersection matrix of D_0 is negative definite, by Lemma 6 we have that D_∞ is the support of a divisor Y with positive normal bundle. By [8] Theorem 4.2 there exists a birational morphism $f:X\to\tilde{X}\subset\mathbb{P}^\nu$ such that f is an isomorphism in a Zariski open neighborhood of D_∞ and $af(Y)$ for some big positive integer a is a hyperplane section. We have $f=[f_0:f_1:\ldots:f_\nu]$, where $f_0,f_1,\ldots,f_\nu$ is a $\mathbb{C}$-basis of $H^0(X,\mathcal{O}_X(aY))$ for $a>0$ big enough. Here $\mathcal{O}_X(aY)$ is the sheaf of meromorphic functions u on X with $div(u)+aY>0$. Since $\mathbb{C}^*$ acts on $H^0(X,\mathcal{O}_X(aY))$ we can take f_i's such that $f_i(\lambda(x,t))=t^{q_i}f_i(x)$ for some $q_i\in\mathbb{N}$. It turns out that f is an isomorphism in $X-D_0$ and the divisor D_0 is mapped to a point of $p\in\mathcal{V}$. $\square$

4.2 Existence of a global linearization

We introduce the *attraction basin* B_p of p, by the flow φ, as

$$B_p = \{\varphi(t,z); t \in \mathbb{C}^*; z \in U\},$$

where $U \subset V$ is the image of a neighborhood $\tilde{U}$ of σ_0 in $\tilde{V}$ by the resolution map ρ. The theorem of Suzuki([25]) asserts that the foliation $\mathcal{F}_\varphi$ admits a meromorphic first integral. This implies that the singularities of $\tilde{\mathcal{F}}_\varphi$ are linearizable and together with Theorem 3 that B_p contains an open neighborhood of p. This fact will be proved again during the construction of the conjugacy map between $\mathbb{C}^*$-actions. We aim to construct a conjugacy between φ on B_p and λ on $\mathcal{V}$ establishing the following theorem:

Theorem 5. *The set B_p is an open subset of V and there is a biholomorphism $h : B_p \to \mathcal{V}$ which is a conjugacy between the actions φ and λ, i.e.,*

$$h(\varphi(t,z)) = \lambda(t,h(z)), \text{ for every } (t,z) \in \mathbb{C}^* \times B_p.$$

Proof. It will be enough to show that there is a conjugacy between $\tilde{\varphi}$ on $\tilde{B}_p := \rho^{-1}(B_p)$ and $\tilde{\lambda}$ on $\tilde{\mathcal{V}}$. We start by defining the conjugacy in a neighborhood of σ_0. An immediate consequence of Theorem 4 is that there is a biholomorphic conjugacy $h\colon \tilde{U} \to \tilde{\mathcal{U}}$ between the restrictions of $\tilde{\varphi}$ and $\tilde{\lambda}$, where $\tilde{U}$ is a neighborhood of σ_0 in $\tilde{V}$ and $\tilde{\mathcal{U}}$ is a neighborhood of σ_0 in $\tilde{\mathcal{V}}$. The conjugacy h extends along the flows $\tilde{\varphi}$ and $\tilde{\lambda}$ as follows: For a point $z' \in \tilde{B}_p \backslash D$ there is $t \in \mathbb{C}^*$ with such that $z := \varphi(t,z') \in \tilde{U}$. We define $h(z')$ by the equality $h(z') = \tilde{\lambda}(t^{-1}, h(z))$. It remains to extend h to a neighborhood of the invariant manifolds of the fixed points of $\tilde{\varphi}$ in $\bigcup_{i=1}^{r} \sigma_i$. These points are all simple and lie in the linear chains starting at $r_1, ..., r_s$ in σ_0. Fix the linear chain starting at $r_1{=}p_0$. The linear chain consists of a finite sequence of elements of the divisor $\sigma_0, \sigma_1, ..., \sigma_n$ such that σ_i, for $i \neq 0$ is a Riemann sphere, where the action φ_1 is nontrivial, and $\sigma_i \cap \sigma_{i+1} = \{p_i\}$ is a simple singularity of $\tilde{\varphi}$ for $i = 1, ..., n-1$. Since at each σ_i, $i > 0$, $\tilde{\varphi}$ has two singularities, there is another fixed point of $\tilde{\varphi}$, $p_n \in \sigma_n$. The conjugacy h is already defined on $\sigma_1 \backslash \{p_1\}$. The next lemma will imply that h extends to $\sigma_2 \backslash \{p_2\}$. Proceeding by induction and having already extended h to $\sigma_n \backslash \{p_n\}$ the next lemma will apply again to extend h to the remaining invariant manifold of p_n. The same procedure can be followed on the other linear chains starting at $r_1, ..., r_s$ in σ_0. It only remains to prove the following lemma.

Lemma 7. *Let Z be a holomorphic vector field defined in a neighborhood $\mathcal{N}$ of the origin $0 \in \mathbb{C}^2$ and $Z_1 = nx\frac{\partial}{\partial x} - my\frac{\partial}{\partial y}$ with $n, m \in \mathbb{N}$ be its linear part. Suppose that*

1. *$x = 0$ and $y = 0$ are separatrices of Z*

2. *There is an analytic conjugacy $h\colon \mathcal{N} \setminus \{x = 0\} \to \mathcal{N} \setminus \{x = 0\}$ between Z and Z_1, i.e., $h_*Z = Z_1$.*

Then h extends to $\mathcal{N}$ as an analytic conjugacy between Z and Z_1.

Proof. The vector fields Z and Z_1 are in the *Siegel domain* ([1]), and the axes $\{x = 0\}$ and $\{y = 0\}$ are the only local separatrices for Z and Z_1. The conjugacy h induces a conjugacy between the holonomies of the local separatrices $\{y = 0\}$ and therefore by classical arguments ([12], [10]) the foliations induced by the vector fields Z and Z_1 are analytically equivalent. Let $F\colon \mathcal{N} \to \mathcal{N}$ be a biholomorphism such that $F_*Z = Z_1$. Then the map $G = h \circ F^{-1}\colon \mathcal{N} \setminus \{x = 0\} \to \mathcal{N} \setminus \{x = 0\}$ is a biholomorphism such that $G_*Z_1 = Z_1$. It is then enough to show that such a self-conjugacy for Z_1 extends as a holomorphic self-conjugacy to $\mathcal{N}$. This is proved as follows. Write $G(x, y) = (xu, yv)$ for some holomorphic functions $u(x, y), v(x, y)$ in $\mathcal{N}\backslash\{x = 0\}$. From $G_*Z_1 = Z_1$ we obtain that:

$$nxu_x - myu_y = 0,\ nxv_x - myv_y = 0 \qquad (*)$$

Since G is holomorphic in $\{y = 0\} \setminus \{0\}$ we can write in Laurent series

$$u = \sum_{i\in\mathbb{Z}\, j\in\mathbb{N}} u_{ij}x^i y^j \ ,\ v = \sum_{i\in\mathbb{Z}\, j\in\mathbb{N}} v_{ij}x^i y^j$$

¿From the above relations (*) we obtain

$$(ni - mj)u_{ij} = 0,\ (ni - mj)v_{ij} = 0,\ \forall (i, j) \neq (0, 0).$$

Thus, $u_{ij} = 0$ and $v_{ij} = 0$ if $ni - mj \neq 0$. On the other hand, if $ni - mj = 0$ then $ni = mj \geq 1$ and therefore $i \geq 1$. This shows that the Laurent series above only have positive powers. Therefore G extends as a holomorphic map to the axis $\{x = 0\}$. Since the same argument applies to G^{-1} we conclude that G extends to $\mathcal{N}$ as a biholomorphism preserving Z_1. □

Remark 2. We observe that, as a consequence of Theorem 3 and Lemma 7 the singularity $p \in V$ is *absolutely dicritical* in the sense that there is a neighborhood W of p in V such

that every leaf of $\mathcal{F}$ intersecting W contains a separatrix of $\mathcal{F}$ through p. In other words, for every leaf L of the restriction $\mathcal{F}\big|_W$ the union $L \cup \{p\}$ is a separatrix of $\mathcal{F}$ through p.

5 Basins of attraction of dicritical singularities

The main result of this section is the following.

Theorem 6. *Let φ be a holomorphic action of $\mathbb{C}^*$ on a normal Stein space V of dimension two. If $p \in V$ is a dicritical singularity of $\mathcal{F}_\varphi$ then the attraction basin of p is V. In other words, every orbit of φ on $V \setminus \{p\}$ accumulates on p.*

This theorem will follow from the two lemmas below. A $\mathbb{C}$-action is of type $\mathbb{C}^*$ if its generic orbit is biholomorphic to $\mathbb{C}^*$.

Lemma 8. *Let φ be a holomorphic $\mathbb{C}$-action of type $\mathbb{C}^*$ on a normal Stein space of dimension two V. Suppose that the set of fixed points of φ is discrete and let $p \in V$ be dicritical singularity of $\mathcal{F}_\varphi$. Then the boundary ∂B_p of the basin of attraction $B_p \subset V$ of p is a (possibly empty) union of analytic curves, and each one of these curves accumulates at a nondicritical singularity of φ.*

Proof. Suppose ∂B_p is nonempty. Then it is invariant by $\mathcal{F}_\varphi$, i.e. it is a union of leaves of $\mathcal{F}_\varphi$ and fixed points of φ. We divide the argument in two steps.

Step 1: ∂B_p contains no closed leaf.

Proof of Step 1. Suppose that $L_0 \subset \partial B_p$ is a leaf of $\mathcal{F}_\varphi$. Since $\mathcal{F}_\varphi$ admits a meromorphic first integral, either L_0 is closed in V or it accumulates only on singular points. Suppose that L_0 is closed in V then it is an analytic smooth curve in V. Since V is Stein there is a holomorphic function $h \in \mathcal{O}(V)$ such that $\{h = 0\} = L_0$ in V ([7], Theorem 5, p.99). Since L_0 is a real surface diffeomorphic to a cylinder $S^1 \times \mathbb{R}$, we can take a generator $\gamma\colon S^1 \to L_0$ of the homology of L_0 and a holomorphic one-form α on L_0 such that $\int_\gamma \alpha = 1$. Again because V is Stein by Cartan's lemma there is a holomorphic one-form $\tilde{\alpha}$ on V which extends α. Since $\mathcal{F}_\varphi$ has a meromorphic first integral on V then the holonomy of L_0 is finite, say of order n. Let Σ be a small disc transverse to the leaves of $\mathcal{F}_\varphi$ with $\Sigma \cap L_0 = \{p_0\} \in \gamma(S^1)$. Then there is a fixed power γ_{p_0} of γ which has closed lifts $\tilde{\gamma}_z$ to the leaves L_z of $\mathcal{F}_\varphi$ that contain the points $z \in \Sigma$. Thus, for $z \in \Sigma$ close enough

to p_0 we have $\left|\int_{\tilde{\gamma}_z}\tilde{\alpha}-\int_{\gamma_{p_0}}\tilde{\alpha}\right|<\frac{1}{2}$, but $\gamma_{p_0}=n\gamma$ and, since $\gamma\subset L_0$, $\int_{\gamma_{p_0}}\tilde{\alpha}=n$ so that $\int_{\tilde{\gamma}_z}\tilde{\alpha}\neq 0$. On the other hand $\tilde{\alpha}$ is holomorphic so that $\tilde{\alpha}\big|_{L_z}$ is holomorphic and therefore closed what implies, since $\tilde{\gamma}_z\subset L_z$ is closed, that L_z has nontrivial homology and therefore necessarily $\overline{L}_z\cong\mathbb{C}^*$. However, since $L_0\subset\partial B_p$ there are leaves L_z of $\mathcal{F}$ with $z\in\Sigma$ as above and which satisfy $L_z\subset B_p$. Such a leaf L_z accumulates on p and therefore $L_z\cup\{p\}$ is a holomorphic curve biholomorphic to $\mathbb{C}$ and thus with trivial homology, yielding a contradiction. $\square$

Step 2: ∂B_p contains no isolated singularity, thus it is a union of analytic curves. Each one of these curves contains a nondicritical fixed point of φ.

Proof of Step 2. If ∂B_p contains an isolated point P then $\partial B_p=\{P\}$ and $V=B_p\cup\{P\}$ would be compact contradicting the fact that V is Stein. On the other hand, by the first step each leaf L contained in ∂B_p is not closed in V so that it accumulates at some fixed point P of φ and since $\overline{L}\supset L\cup\{P\}\simeq\mathbb{C}^*\cup\{0\}=\mathbb{C}$ and $\overline{L}$ cannot be compact, it follows that $\overline{L}=L\cup\{P\}$ is an analytic curve in V and L accumulates at no other fixed point of φ. Finally, we observe that if a leaf $L\subset\partial B_p$ accumulates at a fixed point P of φ then this singularity is nondicritical: the basin of attraction of a dicritical singularity in a Stein variety is open and contains an open neighborhood of the singularity. Since $P\in\partial B_p$, then some leaf $L_1\subset B_p$ intersects this neighborhood and therefore L_1 accumulates on both p and P. Such a leaf would be contained in a rational curve in V and this is not possible because V is Stein. Thus P is nondicritical. $\square$

Lemma 9. *Let φ be a holomorphic $\mathbb{C}$-action of type $\mathbb{C}^*$ with a discrete set of fixed points on a normal Stein space of dimension two V and an absolutely dicritical singularity at $p\in V$. Assume that $H^1(B_p,\mathbb{R})=0$ (for instance, if B_p is simply-connected). Then $V=B_p$.*

Proof. Let us first prove that $V=B_p\cup\partial B_p$. Indeed, put $A=V-\partial B_p$ and $B=B_p$. Then A is a connected open subset of V because by Lemma 8, ∂B_p is a thin set and therefore it does not disconnect V. Since B is also open and connected and $\partial A=\partial B$ it follows that $A=B$ because $B\subset A$. Therefore $V=B_p\cup\partial B_p$. Let us now prove that $\partial B_p=\emptyset$. Suppose there is some analytic curve (leaf) $L_0\subset\partial B_p$. Again, since V is Stein there is a holomorphic function $f\in\mathcal{O}(V)$ such that $\{f=0\}=\overline{L_0}$ in V ([7], Theorem 5,

p.99). Define the meromorphic one-form $\alpha = \frac{df}{f}$ on V, the polar set of α is $\overline{L_0}$. Given a disc $\Sigma \cong \mathbb{D}$ transverse to the leaves of $\mathcal{F}_\varphi$ with $\Sigma \cap L_0 = \{p_0\}$ we consider a simple loop $\gamma\colon S^1 \to \Sigma$ around $p_0 \in \Sigma$ such that $\int_\gamma \alpha = 2\pi\sqrt{-1}$. We can assume that $\gamma(S^1) \subset B_p$ because $\Sigma \setminus (\Sigma \cap \partial B_p) \subset B_p$ and ∂B_p is thin. Since by hypothesis $H^1(B_p, \mathbb{R}) = 0$ it follows that $\int_\gamma \alpha = 0$ yielding a contradiction. Therefore ∂B_p contains no leaf. Thus $V = B_p$ and the Lemma is proved. $\square$

Proof of Theorem 6. Since V is Stein, all regular leaves of $\mathcal{F}_\varphi$ are biholomorphic to $\mathbb{C}^*$. By Proposition 5 and Theorem 2 of [20] and the fact that $\mathcal{F}_\varphi$ has a dicritical singularity, φ has isolated fixed points. In view of Lemmas 8 and 9 it is enough to observe that $H^1(B_p, \mathbb{R}) = 0$. This is clear since the basin B_p of the action φ is biholomorphic to the basin $\mathcal{V}$ of the linear periodic flow λ on $\mathcal{V}$, and moreover $H^1(\mathcal{V}, \mathbb{R}) = 0$. $\square$

Remark 3. The proof of Theorem 6 also shows that $V \setminus \{p\}$ contains no singularity of φ which is dicritical as a singularity of $\mathcal{F}_\varphi$.

Proof. Suppose by contradiction that q is a dicritical singularity of φ then we consider the attraction basin B_q of q and proceeding as for p we prove that $V \setminus \partial B_q = B_q$. On the other hand, since ∂B_q is a thin set we have that $V \setminus \partial B_q$ is connected. Clearly we have $\partial B_p \cap B_q = \emptyset$ because otherwise, since B_q is open, there would be orbits contained in B_p and B_q, which is not possible because these orbits would be contained in rational curves. Analogously we have $\partial B_q \cap B_p = \emptyset$. Thus, the only possibility is to have $\partial B_p = \partial B_q$. Therefore, $V \setminus \partial B_p = V \setminus \partial B_q$, i.e, $B_p = B_q$ and this gives a contradiction.

$\square$

6 Proof of the main theorems

Proof of Theorem 1. Let us be given a pair (V, φ) as in Theorem 1. By Theorem 6 the basin of attraction of the dicritical singularity p is the whole space V, i.e $B_p = V$. By Theorem 5 there is a biholomorphic conjugacy $h\colon V \to \mathcal{V}$ between φ and λ. Finally, by Proposition 1 the variety $\mathcal{V}$ is affine and the the action λ of $\mathbb{C}^*$ on $\mathcal{V}$ in some affine coordinates is good. Note that our proof gives an alternative proof of Proposition 1.1.3, page 207 in [14]. $\square$

Proof of Theorem 2. Each data in Theorem 2 gives us a linear model variety $(\mathcal{V}, \lambda)$ and in a similar way as in Theorem 5 we can prove that two linear models are biholomorphic if and only their correspondings data are the same. On the other hand, we have proved that each pair (V, φ) as in Theorem 1 is biholomorphic to a linear model variety $(\mathcal{V}, \lambda)$. $\square$

Proof of Corollary 1. Since V is smooth, the resolution process of $p \in V$ is the blow-up resolution ([17]) for the foliation $\mathcal{F}_\varphi$ at p. In particular, σ_0 is a negatively embedded projective line. Thus, by Theorem 1 there is a good action σ_Q on $\mathcal{V} \subset \mathbb{C}^{n+1}$ equivalent to φ on V. Thus $\mathcal{V}$ is a quasi-homogeneous non-singular algebraic surface on $\mathbb{C}^{n+1}$ and therefore it is a graph, hence equivalent to an affine plane by algebraic change of coordinates. $\square$

Remark 4. A *quasi-homogeneous surface singularity* (see for instance [18] Chapter III, page 67) is a 2-dimensional analytic variety $V \subset \mathbb{C}^m$ with an isolated singularity at $0 \in \mathbb{C}^m$ supporting a $\mathbb{C}^*$-action φ which is *good* in the sense that every non-singular orbit accumulates (only) at $0 \in \mathbb{C}^m$. As a consequence of our Theorem 1 we obtain that if V is a two-dimensional Stein space with a $\mathbb{C}^*$-action having a dicritical singularity at $p \in V$ then $p \in V$ is a quasi-homogeneous surface singularity.

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